ON THE STABILITY OF PERIODIC MOTIONS

(OB USTOICHIVOSTI PERIODICHESKIKH DVIZHENII)

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In what follows the stability of the periodic solution $z = z^0(t)$ with period τ of the system of differential equations

$$\frac{dz}{dt} = f(z, t) \qquad (f(z, t + \tau) \equiv f(z,t)) \tag{0.1}$$

is investigated in ordinary (nonsingular) cases (z is an n-dimensional vector column).

In the well-known studies by Liapunov and Poincare and in numerous subsequent investigations it is assumed that the right-hand sides of (0.1) are continuous and can be represented in the form of sums of linear terms and nonlinear remainders. Here we consider a more general "discontinuity" type, when the surfaces of discontinuity

$$F_\alpha(z, t) = 0 \qquad (\alpha = 1, 2, \ldots) \tag{0.2}$$

are given, and the right-hand sides $f(z, t)$ of equations (0.1) can have discontinuities on these surfaces. The restrictions put on the functions $f(z, t)$ and $F_\alpha(z, t)$ are the same as in the papers [1,2].

This paper shows that Liapunov's theorem on the linear approximation, and the Andronov and Vitt theorem on the possibility of not taking the unit root of the characteristic equation into account when investigating the autonomous case, can be generalized to the systems of more general type considered here. Liapunov's direct method is replaced by the method of point transformations, and instead of representing the function $f(z,t)$ as a sum of linear terms and a nonlinear remainder, the variational equations are used.

In the classical continuous case the linear approximation and the variational equations coincide. In the "discontinuous" case under consideration, the evaluation of variations, including the discontinuities, leads to linear relations, differential or algebraic, which in the discontinuous case act as a linear approximation. The corresponding theorem, analogous to Liapunov's, was proved by his direct method [2]. It is not immediately clear, however, how a theorem analogous to that of Andronov and Vitt can be proved by the same method for the discontinuous case. On the other hand, the use of the method of point transformations has enabled us to avoid this difficulty and made it possible for us to expound the whole theory of stability of periodic processes of discontinuous systems both for the autonomous and the nonautonomous cases.

The idea of the possibility of exploiting the method of point transformations for our purpose arose after we had seen the manuscript of the study by Neimark, which has since been published [3]. This paper contains a comprehensive study of the problems of stability by the method of point transformations, but it does not contain the equations for the linear approximation of the system (0.1) in the discontinuous case (see below, equations (3.1) + (3.4). As a result, for the discontinuous case paper [3] does not contain the proofs of Liapunov's theorem or that of Andronov and Vitt in the form in which these theorems are formulated and proved for the continuous case.

1. Connection with a point transformation. The periodic solution $z^0(t)$ determines a closed curve, called a cycle, in the z-space, For the origin O of the coordinates take the initial point $z^0(0)$ on this cycle, i.e. put $z^0(0) = 0$. Then also $z^0(\tau) = 0$. Let $y = z(0)$ be the initial deviation*, and $z = \phi(t, y)$ the corresponding solution of the system (0.1). Then the integral curves $z = \phi(t, y)$ determine the point transformation

$$y^* = g(y) \qquad (g(y) \equiv \varphi(\tau, y)) \tag{1.1}$$

for which the deviation y is the transform of the initial deviation y^*, corresponding to the instant $t = \tau$.

The point $y = 0$ is a fixed point of transformation (1.1), which transforms a certain neighborhood of this point into a neighborhood of the same point again. Together with transformation (1.1), we shall consider the iterated transformations $y_m{}^* = g_m(y)$, where $g_m(y) = g[g_{m-1}(y)]$ ($m = 1, 2, \ldots$; $g_0(y) \equiv y$).

Let us introduce certain definitions. The fixed point $y = 0$ of the transformation $y^* = g(y)$ is said to be stable if for arbitrary $\epsilon > 0$ the

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* Let us take the initial instant of the time $t = 0$ different from the instants t_α at which the integral curve $z = z^0(t)$ intersects the discontinuity surfaces $F_\alpha(z, t) = 0$ ($\alpha = 1, 2, \ldots$).

inequality* $|g_m(y)| < \epsilon$ holds for all $m = 0, 1, 2, \ldots$, provided $|y| < \delta = \delta(\epsilon)$. If, in addition, $\lim g_m(y) = 0$ as $m \to \infty$, provided $|y| < \delta_0$ (where δ_0 denotes a certain fixed number), then the fixed point $y = 0$ of the transformation $y^* = g(y)$ is said to be asymptotically stable.**

Let us note that from the inequalities

$$|\varphi(t, y) - \varphi(t, 0)| < \varepsilon \qquad (t \geqslant 0,\ |y| < \delta(\varepsilon))$$

expressing the stability of the periodic solution $z^0(t) = \phi(t, 0)$, provided we restrict ourselves to discrete instants $t = m\tau$, there at once follows the stability of the fixed point $y = 0$. Also conversely, from the stability of the fixed point there follows the stability of the periodic solution of the system (0.1). This can easily be seen from the identity[x]

$$\varphi(t,y) = \varphi(t', g_m(y)) \qquad (g_m(y) = \varphi(m\tau, y), \quad t' = t - m\tau) \tag{1.2}$$

in which m is a nonnegative integer. If m is determined from the inequalities $m\tau \leqslant t < (m+1)\tau$, then t' varies in the finite interval $0 \leqslant t' < \tau$, and the stability follows at once from the theorem on the continuous dependence of solutions on the initial conditions throughout a finite time interval. The situation is the same in the case of asymptotic stability. Thus, according to Liapunov, the periodic solution $z^0(t)$ of the system of differential equations (0.1) is stable (asymptotically stable) only when the fixed point $y = 0$ of the point transformation $y^* = \phi(\tau, y)$ is stable (asymptotically stable).

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* For an n-dimensional vector $y(y_1, \ldots, y_n)$ its modulus $|y|$ is defined as the "Euclidean length"

$$|y| = \sqrt{|y_1|^2 + \cdots + |y_n|^2}$$

** The limit transition $m \to \infty$ is fulfilled uniformly with respect to $y(|y| < \delta_0)$. This follows from the stability of the fixed point $y = 0$.

x The identity (1.2) follows from the periodicity of the right-hand sides of the system (0.1) with respect to t (with the period τ).

2. Linearization of a point transformation. The point transformation $y^* = g(y)$, where $g(y) = \phi(r, y)$, is differentiable* for $y = 0$, i.e. the Jacobian matrix $J = (\partial g/\partial y)_{y=0}$ exists and

$$g(y) = Jy + o(|y|) \qquad \left(\frac{o(|y|)}{|y|} \to 0 \text{ for } y \to 0\right) \tag{2.1}$$

Let $\mu = \min |\lambda(J)|$ and $\nu = \max |\lambda(J)|$ be the smallest and the largest of the moduli of the characteristic numbers of the matrix** J. The following then holds:

1. If in a certain neighborhood of the fixed point $y = 0$ all the $g_m(y)$ exist[x], then in a sufficiently small neighborhood of this point the inequalities

$$K(\mu - \varepsilon)^m |y| \leqslant |g_m(y)| \leqslant L(\nu + \varepsilon)^m |y| \qquad (m = 1, 2, \ldots) \tag{2.2}$$

are fulfilled, and for two given points $y' \neq 0$ and $y'' \neq 0$ we have

$$|g_m(y')| \leqslant K_1(\mu + \varepsilon)^m |y'|, \quad |g_m(y'')| \geqslant L_1(\nu - \varepsilon)^m |y''| \quad (m = 1, 2, \ldots) \tag{2.3}$$

where ϵ is an arbitrarily small positive number and K, L, K_1, L_1 are positive constants which do not depend on m.

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* In fact, in the (z, t)- space a small region Q in the plane $t = 0$, containing the point $y = 0$ in its interior, is transformed by means of the integral curves into a region Q^* in the plane $t = r$. Between the planes $t = 0$ and $t = r$ the integral curves intersect the surfaces of discontinuity $F_\alpha(z, t) = 0$ $(\alpha = 1, \ldots, s)$, mapping the region Q into regions Q_α $(\alpha = 1, \ldots, s)$, lying on these surfaces. In this way the transformation $Q \to Q^*$ decomposes into transformations $Q \to Q_1$, $Q_1 \to Q_2$, $\ldots$, $Q_s \to Q^*$. Each of these transformations is differentiable at the point lying on the curve $z^0(t)$. Therefore the transformation $Q \to Q^*$, i.e. $y^* = g(y)$ is also differentiable at $y = 0$. Here an essential use is made of the property of smoothness of the discontinuity surface $F_\alpha(z, t) = 0$ at the point of intersection M_α of the integral curve $z = z^0(t)$ with the surface of discontinuity $F_\alpha(z, t) = 0$ $(\alpha = 1, 2, \ldots)$.

** The transformation $y^* = \phi(r, y)$ is reversible; therefore the Jacobian $|J| \neq 0$, and consequently $\mu > 0$.

x This condition is always satisfied if the fixed point is stable, i.e. for all successive iterations, all the $g_m(y)$ lie in the region of definition of the function $g(y)$.

2. If $\nu < 1$, then in a certain neighborhood of the fixed point $y = 0$ all the $g_m(y)$ are significant, and therefore the inequalities (2.2) and (2.3) are fulfilled.

For the proof of properties 1 and 2 we will make use of the following lemma.

Lemma. If A is an $n \times n$ matrix and

$$|\lambda_i(A)| < k \; (i = 1, \ldots, n) \qquad \left(\text{or} \quad |\lambda_i(A)| > k \quad (i = 1, \ldots, n)\right)$$

then there exists a similar matrix $B = T^{-1}AT$, the "norm" of which satisfies the inequality

$$\|B\| = \max_{x \neq 0} \frac{|Bx|}{|x|} < k \qquad \left(\text{or} \quad \|I\|^- = \min_{x \neq 0} \frac{|Bx|}{|x|} > k \right)$$

If A is a real matrix, then the matrix B may also be selected as a real matrix.

In fact, let $B = T^{-1}AT$ be a triangular matrix with characteristic numbers λ_i along the main diagonal and with small non-diagonal elements γ_{ij}, i.e. all $|\gamma_{ij}| < \eta$, where η is arbitrarily small. Then for an arbitrary column z we have

$$|Bz| \leqslant [\max|\lambda_i| + (n-1)\eta]\,|z| < (k - \eta)\,|z|$$

Hence

$$\|B\| < k$$

If B is a complex triangular matrix, then it can be replaced by a similar real matrix $U^{-1}BU$, since the basis in the space to which the matrix B refers can be selected in such a way that besides the complex vector $e = (1/\sqrt{2})(p + iq)$, the complex conjugate vector $e' = (1/\sqrt{2})(p - iq)$ is also present. Then the transition from the complex base e, e' to the real base $p, q, \ldots$ is realized by means of a unitary matrix U. Here $\|U^{-1}BU\| = \|B\| < k$. The case when all $|\lambda_i| > k$ is analogously investigated. Hence the Lemma is proved.

Let all the $g_m(y)$ be significant. In accordance with the Lemma (for $k = \nu + \epsilon$) select $B = T^{-1}JT$ and make the transformation of variables $y = Tz$. Then

$$z^* = f(z), \qquad f(z) = Bz + o\,(|z|)$$

and, consequently, whenever $\|B\| < k$, we have for small $|z|$

$$|f(z)| \leqslant k|z| \qquad \text{и} \quad |f_m(z)| \leqslant k^m\,|z| \qquad (m = 1, 2, \ldots)$$

But $g(y) = Tf(T^{-1}y)$ and, in general, $g_m(y) = Tf_m(T^{-1}y)$.

Therefore

$$|g_m(y)| \leqslant \|T\| \, \|T^{-1}\| \, k^m |y|$$

i.e. the right-hand inequalities of (2.2) hold for $L = \|T\| \, \|T^{-1}\|$ and $k = \nu + \epsilon$. In particular, if $\nu < 1$, then for small $\epsilon > 0$ also $k = \nu + \epsilon < 1$. Then from $|f(z)| \leqslant k|z|$ it follows that all the $f_m(z)$, and hence all the $g_m(y)$ are significant in a small neighborhood of the fixed point.

Now, for $\nu > 0$, let

$$A = T^{-1}BT, \qquad B = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$$

where all the moduli of the characteristic numbers of the matrix C are equal to ν but those of the matrix D are less than ν. In accordance with the Lemma it can be assumed that $\|C\|^- > \nu - \epsilon$ and $\|D\| < \nu - 3\epsilon$ for a small $\epsilon > 0$. Then, after the transformation of the variables $y = Tz$, we have $z^* = f(z)$, where

$$f(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = Bz + o(|z|), \qquad z = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$f_1(z) = Cu + o(|z|), \qquad f_2(z) = Dv + o(|z|)$$

Introduce the notations

$$\{z\} = |u| - |v|, \qquad \{f(z)\} = |f_1(z)| - |f_2(z)|$$

Then whenever $|z| > |u| > \{z\}$, we have

$$\{f(z)\} - (\nu - 2\varepsilon)\{z\} \geqslant [|Cu| - (\nu - \varepsilon)|u|] - [|Dv| - (\nu - 3\varepsilon)|v|]$$

Hence on the basis of the inequalities $\|D\|^- > \nu - \epsilon$, $\|D\| < \nu - 3\epsilon$ it follows that in a small neighborhood of the fixed point $z = 0$ the right-hand side of the last inequality is nonnegative. Therefore

$$|f(z)| \geqslant \{f(z)\} \geqslant (\nu - 2\varepsilon)\{z\}$$

and, in general

$$|f_m(z)| \geqslant \{f_m(z)\} \geqslant (\nu - 2\varepsilon)^m \{z\} \qquad (m = 1, 2, \ldots)$$

Now let $z'' = u'' \neq 0$, $v'' = 0$. Then $\|z''\| = \{z''\} > 0$ and

$$|f_m(z'')| \geqslant (\nu - 2\varepsilon)^m |z''|, \quad |f_m(z'')| = |T^{-1} g_m(y'')| \leqslant |T^{-1}\| \, |g_m(y'')|, \quad |y''| \leqslant \|T\| \, |z''|$$

Putting $L_1 = (\|T\| \, \|T^{-1}\|)^{-1}$ and replacing 2ϵ by ϵ, we obtain the right-hand inequality of (2.3). The left-hand inequalities (2.2) and (2.3) are obtained at once, if the right-hand inequalities (2.2) and (2.3) are applied to the inverse transformation

$$y = I^{-1} y^* + o(|y^*|)$$

The validity of inequalities (2.2) and (2.3) is thus proved. The inequality (2.2) will be used later, in Section 5. Let us note here that from the inequalities (2.2) and (2.3) there at once follows the known criterion of stability for the fixed point*: the fixed point $y = 0$ of the point transformation $y^* = g(y)$ is asymptotically stable if $\nu < 1$, and unstable if $\nu > 1$.

3. Variational equations. Let $z(t, \mu)$ be an arbitrary one-parameter family of solutions of the system (0.1), reducing to $z^0(t)$ for $\mu = 0$. Let the function $z(t, \mu)$ be differentiable for $\mu = 0$ and $t \geqslant 0$. Denote by x the first variation** of the solution $z^0(t)$ of the system (0.1):

$$x = \delta z = \left(\frac{\partial z(t,\mu)}{\partial \mu}\right)_{\mu=0} \delta\mu$$

If we replace z by $z(t, \mu)$ in (0.1) and next differentiate term-by-term with respect to μ and put $\mu = 0$, then we obtain a linear system of differential equations with periodic coefficients

$$\frac{dx}{dt} = \left(\frac{\partial f}{\partial z}\right)_{z=z^0(t)} x \tag{3.1}$$

which is satisfied by the variation $x(t)$ in each of the intervals

$$t_{\alpha-1} \leqslant t \leqslant t_\alpha \qquad (\alpha = 1, \ldots; \ t_0 = 0)$$

Inasmuch as the integral curve $z^0(t)$ at the instants t_a has breaks, the variation $x(t)$ for $t = t_a$ has discontinuities. Let us calculate the magnitudes of these discontinuities. By $t(\mu)$ denote the instant corresponding to the intersection of the curve $z(t, \mu)$ with the surface $F_a(z, t) = 0$, so that $t(0) = t_a$. Then the point $z[t(\mu), \mu]$ for arbitrary μ is situated on the surface $F_a = 0$. The differential of this function $z[t(\mu), \mu]$ evaluated at $\mu = 0$, and calculated for the approach from the region H_a or from the region H_{a-1}, is equal to[x]

$$\delta z = f^+{}_\alpha \delta t(\mu) + x^+{}_\alpha = f^-{}_\alpha \delta t(\mu) + x^-{}_\alpha$$

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* Another proof of the criterion by means of the Liapunov functions is contained in paper [3].

** In what follows it will be convenient to understand by the variation δz the derivative $(\partial z/\partial\mu)_{\mu=0}$, i.e. to assume $\delta\mu = 1$.

x Here $f^+{}_a$, $x^+{}_a$ (and correspondingly $f^-{}_a$, $x^-{}_a$) denote the value of the function $f(z, t)$ at the point of the integral curve $z^0(t)$ and the value of the variation $x(t)$ for $t = t_a + 0$ (correspondingly for $t = t_a - 0$). The regions H_{a-1} and H_a adjoin the surface of discontinuity in a neighborhood of the point M_a. In each of these regions the function $f(z, t)$ is continuous.

Hence

$$x^{+}_{\alpha} - x^{-}_{\alpha} = -\xi_{\alpha}\delta t(\mu) \qquad (\xi_{\alpha} = f^{+}_{\alpha} - f^{-}_{\alpha}) \tag{3.2}$$

On the other hand, differentiating term-by-term (for $\mu = 0$) the identity $F_{\alpha}\{z[t(\mu), \mu], t(\mu)\} = 0$, we obtain

$$\left(\frac{dF_{\alpha}}{dt}\right)^{\pm} \delta t(\mu) + \left(\frac{\partial F_{\alpha}}{\partial z}\right)_{M_{\alpha}} x_{\alpha}^{\pm} = 0 \tag{3.3}$$

where $(dF_{\alpha}/dt)^{\pm}$ denotes the total derivative of $F_{\alpha}[z^{0}(t), t]$ for $t = t_{\alpha} \pm 0$, and $(\partial F/\partial z)$ denotes the column composed of $(\partial F_{\alpha}/\partial z_{i})$ $(i = 1, \ldots, n)$. Hence determining $\delta t(\mu)$ and substituting into (3.2), we finally obtain*

$$x^{+}_{\alpha} = S_{\alpha}x^{-}_{\alpha} \qquad (\alpha = 1, 2, \ldots) \tag{3.4}$$

where the constant $n \times n$ matrices S_{α} are determined by the equalities

$$S_{\alpha} = E + \xi_{\alpha}h_{\alpha}^{-} = (E - \xi_{\alpha}h^{+}_{\alpha})^{-1}, \qquad h_{\alpha}^{\pm} = \frac{(\partial F_{\alpha}/\partial z)_{m_{\alpha}}}{(dF_{\alpha}/dt)^{\pm}} \quad (\alpha = 1, 2, \ldots) \tag{3.5}$$

Here E denotes the unit matrix.

The linear system (3.1) + (3.4) will be called the variational equations for the solution $z^{0}(t)$ of the system (1.1). When $f(z, t)$ is a continuous function, the discontinuity conditions (3.4) are absent and there remain only the usual variational equations, which hold for all $t \geqslant 0$.

4. Extension of Liapunov's theorem on stability of periodic solutions to the "discontinuous" case. Consider an $n \times n$ matrix $X(t)$, the columns of which are the solutions of the equations of variations (3.1) + (3.4). The determinant $|X(t)|$ satisfies the formula (a generalization of the Jacobian formula for the "continuous" case)

$$|X(t)| = |X(0)|\,|S_1| \ldots |S_{\alpha-1}| \exp\int_0^t \mathrm{Sp}P(t)\,dt \qquad \begin{pmatrix} t_{\alpha-1} \leqslant t \leqslant t_{\alpha} \\ \alpha = 1, 2, \ldots; t_0 = 0 \end{pmatrix} \tag{4.1}$$

Here $P() = (\partial f/\partial z)_{z = z^{0}(t)}$ is the matrix of the coefficients of the linear system of differential equations (3.1)**.

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* The conditions of discontinuity (3.4) and (3.5) were obtained differently in papers [1,2].

** Formula (4.1) is obtained if the usual Jacobian formula is applied to every interval $[t_{\alpha-1}, t_{\alpha}]$ and the relations (3.4) are taken into account.

Since, according to (3.5), each matrix S_a is nonsingular, i.e. $|S_a| \neq 0$, the matrix $|X(t)|$ is different from zero for all $t > 0$, provided it is different from zero for $t = 0$. In such a case the matrix $X(t)$ is a fundamental matrix, i.e. its columns are made up of n linearly independent solutions of the system (3.1) + (3.4). Any other fundamental matrix can be represented in the form $X(t)C$, where C is an arbitrary nonsingular constant $n \times n$ matrix. Because of the periodicity of the system (3.1) + (3.4), not only the matrix $X(t)$ but also the matrix $X(t + \tau)$ is a fundamental matrix. Therefore

$$X(t + \tau) = X(t)U \tag{4.2}$$

where U is a nonsingular constant matrix. The matrix U is determined by the system (3.1) + (3.4) to within a transformation*.

As in the "continuous" case, the equation $|U - \lambda E| = 0$ is called the characteristic equation of the system (0.1). We will prove that one of the matrices U is the Jacobian matrix $J = (\partial\phi(\tau, y)/\partial y)_{y=0}$. In fact, the columns of the matrix $X(t) = (\partial\phi(t, y)/\partial y)_{y=0}$ are the variations**, which, as has been shown, satisfy the system (3.1) + (3.4). Moreover, $X(0) = (\partial\phi(0, y)/\partial y)_{y=0} = \partial y/\partial y = E$. Consequently, $|X(0)| = 1$ and $X(t)$ is a fundamental matrix. Substituting $t = 0$ in the identity (4.2) and making use of $X(0) = E$, we obtain

$$U = X(\tau) = \left(\frac{\partial\varphi(\tau, y)}{\partial y}\right)_{y=0} = \left(\frac{\partial g}{\partial y}\right)_{y=0} = J \tag{4.3}$$

The characteristic numbers of the matrix J thus coincide with the roots of the characteristic equation of the system (0.1). Therefore, the criteria on stability of a fixed point mentioned in Section 2 can immediately be applied to the variational equations (3.1) + (3.4). This at once leads to the following theorem, analogous to Liapunov's theorem on the stability of the periodic motion in the continuous case.

Theorem 1. If ν is the maximum modulus of the roots of the characteristic equation $|U - \lambda E| = 0$, then the periodic solution $z^0(t)$ of the system (0.1) is asymptotically stable for $\nu < 1$ and unstable for $\nu > 1$.

5. **Autonomous systems.** If system (0.1) is autonomous, i.e. $f(z, t)$ does not depend explicitly on t, so that $(\partial f/\partial t) = 0$, and the

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* If instead of $X(t)$ the matrix $X(t)C$ is taken for a fundamental matrix, then the matrix U is to be replaced by $C^{-1}UC$.

** In the k-th column of the matrix $(\partial\phi(t, y)/\partial y)_{y=0}$ appear the elements $(\partial\phi(t, y)/\partial y_k)_{y=0}$, evaluated for the zero values of all y_j.

equations of the discontinuity surfaces $F_a = 0 (a = 1, 2, \ldots)$ also do not explicitly contain the time t, then the system (0.1) admits a family of periodic solutions $z^0(t + \mu)$, and therefore the vector velocity

$$\dot{z}^{\circ}(t) = \left(\frac{\partial z^{\circ}(t+\mu)}{\partial \mu}\right)_{\mu=0}$$

satisfies the variational equations (3.1) + (3.4). Inasmuch as the variational equations possess a periodic solution $z^0(t)$, then, as in the "continuous" case, the characteristic equation $|U - \lambda E| = 0$ has a unit root. Therefore, the criterion for asymptotic stability is not applicable to the autonomous case. We will show that the theorem of Andronov and Vitt [4], established for the "continuous" case, can also be extended to the "discontinuous" case, provided that by the variational equations we understand the combined system (3.1) + (3.4).

Consider an n-dimensional z-space. Without loss of generality in our deductions, assume* that $z^0(0) = 0$, $z_1{}^0(0) = 1$, $z_2{}^0(0) = \ldots = z_n{}^0(0) = 0$. Choosing the initial point y (with the coordinates $y_2, \ldots, y_n$) in the hyperplane $z_1 = 0$, denote by $\phi(t, y)$ the corresponding solution of the system (0.1). Construct the matrix**

$$X(t) = \left\| z^{\circ}(t), \left(\frac{\partial \varphi}{\partial y_2}\right)_{y=0}, \ldots, \left(\frac{\partial \varphi}{\partial y_n}\right)_{y=0} \right\| \tag{5.1}$$

This matrix turns out to be a normalized fundamental matrix for the variational equations, since the columns of (5.1) satisfy the system (3.1) + (3.4) and $X(0) = E$. Therefore [see (4.3)]

$$U = \begin{pmatrix} 1 & * \\ 0 & V \end{pmatrix} \tag{5.2}$$

where $V = (\partial \Phi / \partial y)_{\substack{y=0 \\ t=\tau}}$ and Φ is an $(n-1)$ dimensional vector column with the coordinates $\phi_2, \ldots, \phi_n$.

Inasmuch as in the z-phase space the integral curve $z^0(t)$ intersects the hyperplane $z_1 = 0$ at t h $m\tau$ $(m = 1, 2, \ldots)$, also a close integral curve $z = \phi(t, y)$ (for $|y|$ small) will also intersect this hyperplane

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* Assume that the velocity $z^0(0)$ is directed along the z_1-axis, and by a proper selection of the scale make the modulus of the velocity equal to one. For the hyperplane $z_1 = 0$ we can take any arbitrary hyperplane which intersects the cycle $z^0(t)$ at O.

** On the right-hand side of (5.1) are indicated the columns which constitute the matrix.

for the values $t = t_m(y)$, where $t_m(y)$ are the roots of the equation $\phi_1(t, y) = 0 (m = 1, 2, \ldots, t_m(0) = m\tau)$.

Consider the point transformation

$$y^* = k(y) \qquad (k(y) = \varphi(t_1(y), y)) \tag{5.3}$$

realized by the integral curves and transforming a neighborhood of O in the hyperplane $z_1 = 0$ into another such neighborhood. The curve $z^0(t) = \phi(t, 0)$ determines the fixed point $y = 0$ of this transformation. The transformation (5.3) (to distinguish it from the transformation (1.1)) will be called the truncated point transformation.

Let us compute the Jacobian matrix of the truncated point transformation:

$$J' = \left(\frac{\partial k}{\partial y}\right)_{y=0} = \left(\frac{\partial}{\partial y}\Phi[t_1(y), y]\right)_{y=0} = \left(\frac{\partial \Phi(\tau, y)}{\partial y}\right)_{y=0} + \left(\frac{\partial \Phi(t, 0)}{\partial t}\right)_{t=0}\left(\frac{\partial t_1(y)}{\partial y}\right)_{y=0}$$

Since $V = (\partial \Phi(\tau, y)/\partial y)_{y=0}$, and the "truncated" initial velocity $(\partial \Phi(t, 0)/\partial t)_{t=0}$, according to assumption, is zero, then $J' = V$.

Thus, owing to the criterion of stability of the fixed point and (5.2), the following proposition holds.

If, besides the unit root, the moduli of the remaining $n - 1$ roots of the characteristic equation $|U - \lambda E| = 0$ are all less than unity, then the fixed point of the truncated point transformation (5.3) is asymptotically stable.

The following theorem holds*.

Theorem 2. If, besides the unit root, the moduli of all the remaining $n - 1$ roots of the characteristic equation are less than one, then the periodic solution $z^0(t)$ of the system is stable according to Liapunov. Moreover, it is "asymptotically stable to within the phase", i.e. for every solution $z(t)$ of the system (0.1) which is close to $z^0(t)$($|z(0) - z^0(0)| < \delta_0$, where δ_0 is a given number), there exists an α which depends continuously on $z(0)$ such that

$$\lim[z(t+\alpha) - z^\circ(t)] = 0 \quad \text{for} \quad t \to \infty, \qquad \lim \alpha = 0 \quad \text{for } z(0) \to z^\circ(0) \tag{5.4}$$

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* For the case when $f(z, t)$ is a continuous function, this theorem was established in a somewhat modified form by Andronov and Vitt in 1933 in [4].

Let us first prove the existence of a finite limit:

$$\lim\,[t_m(y) - m\tau] = \beta(y) \quad \text{for } m \to \infty \tag{5.5}$$

For this let us note that

$$t_m(y) - m\tau = \sum_{q-1}^{m} \{t_1[k_{q-1}(y)] - \tau\} \qquad (k_0(y) \equiv y) \tag{5.6}$$

Owing to the differentiability (for $y = 0$) of the function $t_1(y)$ and the inequalities (2.2), we have

$$|\,t_1[k_{q-1}(y)] - \tau\,| \leqslant M\,|\,k_{q-1}(y)\,| \leqslant N(\nu + \varepsilon)^{q-1}|\,y\,| \tag{5.7}$$

where $M > 0$, $N > 0$ are constants, ν is the maximum modulus of the roots of the characteristic equation, different from unity, and $\epsilon > 0$. Assuming that in (5.7) $\nu + \epsilon \lessgtr 1$, we conclude from (5.6) and (5.7) that the difference $t_m(y) - m\tau$ is the m-th partial sum of a uniformly convergent series, and therefore

$$\lim\,[t_m(y) - m\tau] = \beta(y) \qquad \text{for } m \to \infty$$

where $\beta(t)$ is a continuous function which satisfies the inequality

$$|\,\beta(y)\,| \leqslant \Gamma\,|\,y\,| \qquad \left(\Gamma = \frac{N}{1 - \nu - \varepsilon}\right) \tag{5.8}$$

From this there at once follows the asymptotic stability to within the phase of the solution $z^0(t)$ for initial deviations taken in the hyperplane $z_1 = 0$.

In fact, let $m\tau \leqslant t < (m + 1)\tau$. Since the system (0.1) is autonomous, the choice of the initial instant is arbitrary. Therefore, displacing the time by t_m, we have the identity

$$\varphi(t + \beta,\ y) = \varphi(\bar{t},\ \varphi(t_m,\ y)) = \varphi(t,\ k_m(y)) \tag{5.9}$$

where $t = t + \beta - t_m$

The quantity t due to (5.5) varies in a finite interval. Therefore, using the fact that the solutions of the system (0.1) depend continuously on the initial conditions throughout a finite interval, we have

$$\lim\,[\varphi(t,\ k_m(y)) - \varphi(t,\ 0)] = 0 \ \text{ as } \quad m \to \infty \tag{5.10}$$

On the other hand, owing to the periodicity, $\phi(t,\ 0) = \phi(t + m\tau,\ 0)$, and therefore

$$\lim [\varphi(\bar{t},\ 0) - \varphi(t,\ 0)] = 0 \quad \text{as} \quad m \to \infty \tag{5.11}$$

in so far as $t + m\tau - t \to 0$ as $m \to \infty$.

Adding term-by term (5.10) and (5.11), and using (5.9), we obtain

$$\lim [\varphi(t + \beta,\ y) - \varphi(t,\ 0)] = 0 \qquad \text{as} \quad t \to \infty \tag{5.12}$$

This is so, since as $t \to \infty$, so also $m \to \infty$.

Now assume that the initial point $z(0)$ does not lie in the hyperplane $\tau_1 = 0$ and is determined by the equality $z(0) = \phi(\mu,\ y)$, where $|\mu| < \Delta$, $|y| < \Delta\,(\Delta > 0)$. For these inequalities (with a sufficiently small Δ) let the Jacobian be

$$\left|\frac{\partial\varphi}{\partial\mu},\ \frac{\partial\varphi}{\partial y}\right| \neq 0$$

For $\mu = 0$, $y = 0$ this Jacobian is equal to $|X(0)| = 1$ [see (5.1)].

Then all the points $\phi(\mu,\ y)$ for $|\mu| < \Delta_1$, $|y| < \Delta$ cover a certain n-dimensional neighborhood of the point O, and $\phi(\mu,\ y) \to 0$ only as $\mu \to 0$ and $y \to 0$. If $z(0) = \phi(\mu,\ y)$, then $z(t) = \phi(t + \mu,\ y)$. Therefore, in (5.12) replacing $\phi(t + \beta,\ y)$ by $z(t + \beta - \mu)$ and putting $\alpha = \beta - \mu$, we obtain (5.4).

From the reasoning outlined above it is evident that the transition to the limit in the equality (5.4) is realized uniformly with respect to all initial values $z(0)$, satisfying the inequality $|z(0) - z^0(0)| < \delta_0$. From here it easily follows that the solution $z^0(t)$ is stable in Liapunov's sense.

Hence the theorem is proved.

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